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SCATTERING OF SH WAVES BY THIN, SEMI-INFINITE INCLUSIONS

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Abstract—A thin, semi-infinite inclusion is perfectly bonded to a surrounding matrix and subjected to an incident, plane, harmonic SH wave. The scattering field is evaluated to first order by the method of singular perturbations and matched asymptotic expansions, with the thickness-to-wavelength ratio as the perturbation parameter. The inner expansion, valid near the tip of the inclusion, obeys the equations of elastostatics, while outer expansion is a wave field comprising the incident wave along with a scattered field generated by sources along the mid-line of the inclusion and at its tip. The scattered wave is of first order in the perturbation parameter, and its far field, as evaluated by the method of steepest descents, contains non-decaying, reflected and transmitted plane waves as well as a radiated portion decaying as $r^{-1/2}$. For some combinations of incident angle and material properties, there are nulls in the radiated wave at two distinct angles, while in other cases there are none. The angle midway between the nulls is found to be independent of the inclusion-to-matrix density ratio, but to vary with stiffness ratio and angle of incidence. Polar plots of the scattered power flux are given for a 45° incident angle and a range of material properties.

INTRODUCTION

Few problems of elastic wave scattering can be solved explicitly and exactly. In those that can be, the scatterer is inevitably bounded by surfaces of a coordinate system in which the wave operator is separable[1]. Even in these, excepting circular cylindrical and spherical cavities or rigid inclusions, the representations are very cumbersome and not at all transparent. The quest for useful approximate techniques thus continues and indeed escalates, spurred by recent interest and success in practical applications of ultrasonic non-destructive testing.

In this paper we treat the title problem by the technique of singular perturbations and matched asymptotic expansions. This method yields asymptotic formulas for the solution as the ratio of two characteristic lengths—the inclusion thickness and the wavelength—tends to zero. Because the wavelength varies inversely with frequency, the results provide in essence a low-frequency approximation.

Many other workers have undertaken analyses based in some way on the smallness of some physical dimension of the scatterer with respect to wavelength; here we mention those most relevant to the case at hand. Crighton and Leppington[2] solved a problem precisely analogous to the long-wavelength scattering of a plane SH wave by a semi-infinite *cavity* of constant width. They used the same method that will be applied here. There are significant differences, however, in the nature of scattering by long cavities and bonded inclusions. A cavity acts locally as a perfect reflector of incident waves while an inclusion permits some transmission. The cavity permits large offsets across it while the inclusion constrains the relative displacement of points on its opposite surfaces. In the limit of vanishing thickness, a cavity becomes a crack, and still acts as a scatterer, while an inclusion becomes a welded seam between its surfaces and effectively vanishes, leaving a uniform space with no means of scattering an incident wave.

Datta[3, 4] has used the technique of singular perturbations and matched asymptotic expansions to study scattering by ellipsoidal and elliptic cylindrical inclusions with *all* dimensions small compared to a wavelength. The method is particularly well suited to this class of problems, because the inner expansions turn out to be solutions to static elastic problems of remotely stressed solids containing inclusions of the same shape, and for these the results of Eshelby[5] can be used directly.

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Herrerra and Mal[6], and Mal and Herrerra[7], used a perturbation technically to study scattering by thin inclusions. Their method was based on integral equations derived from the Green's function for the unperturbed, infinite space. If applied to a semi-infinite inclusion, it would yield results which agree with the present outer expansion to first order, but would give no information about the actual fields near the tip of the inclusion.

The outer expansion contains singularities situated at the tip of the inclusion. Both the physical significance of these singularities, and the actual state of affairs near the tip of the inclusion, emerge quite clearly when the problem is formulated, as in the present study, from the viewpoint of matched asymptotics. The inner expansion is found to be a *static* elastic field; therefore the actual field near the tip would be that of an elastic inclusion embedded in a remotely, statically strained matrix. Should the inclusion contain angular corners near its tip, there would be stress singularities characteristic of the included angle and material mismatch. If the tip were smooth, there would be no singularities, but there would still be a stress concentration. Although we choose not to pursue the matter here, one could obtain a detailed picture of the stress state near the tip by solving the appropriate static elastic problem for the tip shape in question, with a numerical approach very likely the only recourse. In any event, so long as the inclusion reaches a uniform thickness at a distance from its tip that is small compared with a wavelength, out analysis shows that the outer expansion to first order is independent of the shape of the tip.

The singularities in the *outer* expansion, as found in[6,7]. and the present work, bear no direct connection with the static elastic singularities just mentioned, nor with any actual localized intensification of stresses. Rather, they represent—in the jargon of matched asymptotic expansions—the inner expansion of the outer expansion, and must be directly related, through a "matching principle," to the outer expansion of the inner expansion, i.e. to the far-field of the inner, static elastic solution. The implementation of the matching principle is discussed in detail in the sequel.

FORMULATION

Figure 1 shows the dimensions and coordinates for the problem at hand. The coordinates x and y have been normalized with respect to the characteristic length $\lambda/2\pi \equiv \omega^{-1}(G/\rho)^{1/2}$, where λ is the wavelength of shear waves in the matrix, ω is the frequency, and G is the shear modulus and ρ the density of the matrix. The problem is one of the antiplane shear, or SH waves, where the only nonvanishing displacement component is perpendicular to the $x-y$ plane and is independent of z. A time factor $\exp(-i\omega t)$ is assumed and hereafter suppressed. An inclusion of normalized thickness 2ϵ, shear modulus G', and density ρ' runs along the negative x-axis.

If M and I denote the planar projections of the matrix and inclusion, respectively, the complex displacements must satisfy the wave equations

$$(\nabla^2+1)V=0 \quad [(x,y)\in M], \qquad (\nabla^2+\kappa^2)W=0 \quad [(x,y)\in I], \tag{1}$$

where V and W are the displacements in the matrix and inclusion respectively, $\nabla^2 =$

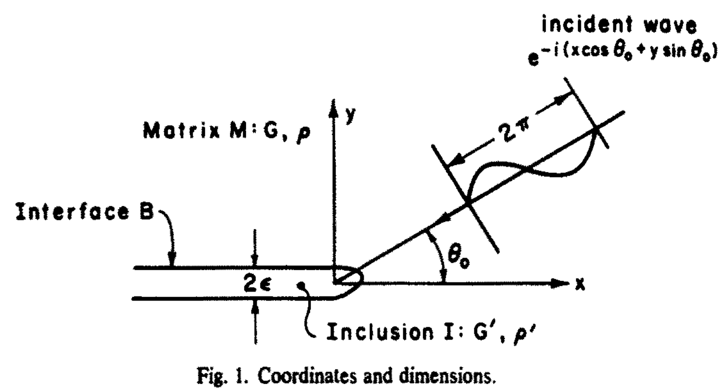

Fig. 1. Coordinates and dimensions.

$\partial^2/\partial x^2 + \partial^2/\partial y^2$, and $\kappa = (G\rho'/G'\rho)^{1/2}$ is the ratio of characteristic wave numbers, or inverse ratio of wavelengths, in the inclusion and matrix.

The inclusion is assumed to be perfectly bonded to the matrix; the necessary continuity of displacements and normal tractions is expressed by

$$V = W, \quad \partial V/\partial n = \mu \partial W/\partial n \qquad [(x,y) \in B], \tag{2}$$

where B is the interface between the matrix M and the inclusion I, n the outer normal to B, and $\mu = G'/G$.

A plane, incident wave of the form

$$v^{(i)}(x,y) = \exp[-i(x\cos\theta_0 + y\sin\theta_0)] \tag{3}$$

impinges on the inclusion. Here θ_0 is the angle of incidence as shown in Fig. 1; we shall restrict it to $(0,\pi/2)$. The unit amplitude signifies that all displacements are normalized with respect to the physical amplitude of the incident wave.

Our object is to find a solution to (1) and (2) subject to the auxiliary conditions that (a) the strain energy density remain bounded in $M \cup I$, and (b) W and $V - v^{(i)}$ represent only outwardly propagating waves. This cannot be done in general, but we shall find the asymptotic form of the solution as ε tends to zero. The result will be strictly valid only in the limit as $\varepsilon \to 0$, but on the basis of experience in other related problems, it can be expected to provide reasonable approximations for small but nonvanishing ε.

OUTER AND INNER ASYMPTOTIC EXPANSIONS

As noted by Van Dyke[8], because ε is the ratio of two characteristic lengths—the half-thickness of the inclusion and $\lambda/2\pi$—no single, uniformaly valid asymptotic expansion is possible. The perturbation is said to be *singular* and the method of singular perturbations and matched asymptotic expansions may be used to find the appropriate expansions.

In preparation, a "stretched" variable $\bar{y}$ is defined by $y = \varepsilon\bar{y}$, and all conditions pertaining to W are restated in terms of a new function $\hat{W}(x,\bar{y};\varepsilon) = \hat{W}(x,y/\varepsilon;\varepsilon) = W(x,y;\varepsilon)$. The reason is that as ε tends to zero the width 2ε of the inclusion shrinks to zero in the $x - y$ plane but remains constant at 2 in the $x - \bar{y}$ plane. Thus from (1) and (2) we have

$$[\partial^2/\partial\bar{y}^2 + \varepsilon^2(\partial^2/\partial x^2 + \kappa^2)]\hat{W} = 0 \qquad [(x,\bar{y}) \in \hat{I}], \tag{4}$$

$$V(x,y) = \hat{W}(x,y/\varepsilon) \qquad [(x,y) \in B], \tag{5}$$

where $\hat{I}$ is the image of I in the $x - \bar{y}$ plane. On the parallel parts of B the normal n is in the y-direction, $\bar{y} = 1$, and the second of (2) becomes

$$\varepsilon\partial V/\partial y = \mu\partial\hat{W}/\partial\bar{y}. \tag{6}$$

Next, we introduce the following tentative "outer expansions"† for V and W:

$$V(x,y;\varepsilon) = \exp[-i(x\cos\theta_0 + y\sin\theta_0)] + \varepsilon v_1(x,y) + o(\varepsilon) \qquad [\varepsilon \to 0, (x,y) \text{ fixed}], \tag{7}$$

$$\hat{W}(x,\bar{y};\varepsilon) = \hat{w}_0(x,\bar{y}) + \varepsilon\hat{w}_1(x,\bar{y}) + \varepsilon^2\hat{w}_2(x,\bar{y}) + o(\varepsilon^2) \qquad [\varepsilon \to 0, (x,\bar{y}) \text{ fixed}]. \tag{8}$$

Notice that the incident wave is the only zero-order term of V; the remainder, representing the scattered wave, vanishes as ε tends to zero.

Now we consider the consequences of the field equations. Substitution of (8) into (4) and

†Asymptotic expansions need not always proceed by powers of ε. Van Dyke[8] describes systematic means of finding the appropriate functions of ε. In the interest of brevity, we *assume* the forms (7) and (8), and take as justification their ability ultimately to meet all the necessary conditions.

repeated application of the limit process $[\varepsilon \to 0, (x,\bar{y})$ fixed$]$ yields

$$\frac{\partial^2 \hat{w}_0}{\partial \bar{y}^2} = \frac{\partial^2 \hat{w}_2}{\partial \bar{y}^2} = \frac{\partial^2 \hat{w}_2}{\partial \bar{y}^2} + \left(\frac{\partial^2}{\partial x^2} + \kappa^2\right)\hat{w}_0 = 0 \quad (-\infty < x < 0, |\bar{y}| < 1). \tag{9}$$

The solutions are

$$\hat{w}_0 = C_0(x) + \bar{y}D_0(x),\ C_1(x) + \bar{y}D_1(x),$$

$$\hat{w}_2 = C_2(x) + \bar{y}D_2(x) - \frac{\bar{y}^2}{2}(C_0'' + \kappa^2 C_0) - \frac{\bar{y}^3}{6}(D_0'' + \kappa^2 D_0), \tag{10}$$

where $C_0(x), \ldots, D_2(x)$ are unknown functions of x for $-\infty < x < 0$, and the primes denote differentiation. The continuity condition (5) obtains at $\bar{y} = y/\varepsilon = \pm 1$, and upon substitution from (7), (8) and (10) yields

$$\exp[-i(x\cos\theta_0 \pm \varepsilon\sin\theta_0)] + \varepsilon v_1(x, \pm\varepsilon) + o(\varepsilon)$$

$$= C_0(x) \pm D_0(x) + \varepsilon[C_1(x) \pm D_1(x)] + o(\varepsilon) \qquad (-\infty < x < 0), \tag{11}$$

where the upper signs are taken together. By expanding the first term for small ε and repeatedly letting ε tend to zero, (11) yields

$$C_0(x) \pm D_0(x) = \exp(-ix\cos\theta_0)$$

$$C_1(x) \pm D_1(x) = \mp i\sin\theta_0 \exp(-ix\cos\theta_0) + v_1(x,0^{\pm}). \tag{12}$$

By forming first the sum, then the difference of each pair in (12) we find

$$C_0(x) = \exp(-ix\cos\theta_0),\ D_0(x) = 0,$$

$$2C_1(x) = v_1(x,0^+) + v_1(x,0^-),$$

$$2D_1(x) = -2i\sin\theta_0 \exp(-ix\cos\theta_0) + v_1(x,0^+) - v_1(x,0^-). \tag{13}$$

The second continuity condition (6) similarly yields

$$\varepsilon\frac{\partial}{\partial y}\{\exp[-i(x\cos\theta_0 + y\sin\theta_0)] + \varepsilon v_1 + o(\varepsilon)\}\Big|_{y=\pm\varepsilon} = \mu\frac{\partial}{\partial\bar{y}}\{C_0 + \bar{y}D_0 + \varepsilon[C_1 + \bar{y}D_1]$$

$$+ \varepsilon^2[C_2 + \bar{y}D_2 - \frac{\bar{y}^2}{2}(C_0'' + \kappa^2 C_0) - \frac{\bar{y}^3}{6}(D_0'' + \kappa^2 D_0)]\}\Big|_{\bar{y}=\pm 1},$$

and this implies

$$\mu D_0 = 0, \qquad \mu D_1 = -i\sin\theta_0 \exp(-ix\cos\theta_0),$$

$$\mu[D_2 \mp (C_0'' + \kappa^2 C_0) - \frac{1}{2}(D_0'' + \kappa^2 D_0)] = \mp\sin\theta_0 \exp(-ix\cos\theta_0) + \frac{\partial v_1}{\partial y}\Big|_{y=0^{\pm}}. \tag{14}$$

We note that the first of (14) is consistent with the second of (13). Substitution of the second of (14) into the fourth of (13) yields

$$v_1(x,0^+) - v_1(x,0^-) = (1 - 1/\mu)2i\sin\theta_0 \exp(-ix\cos\theta_0)H(-x), \tag{15}$$

where $H(x)$ is the Heaviside's unit step function. Similarly, the third of (14) combined with the first and second of (13) yields

$$\partial v_1/\partial y\Big|_{y=0^+} - \partial v_1/\partial y\Big|_{y=0^-} = 2[\sin^2\theta_0 + \mu(\cos^2\theta_0 - \kappa^2)]\exp(-ix\cos\theta_0)H(-x). \tag{16}$$

By the foregoing steps we have derived the jump conditions (15) and (16) on the first order scattered field v_1, and they will turn out to be almost sufficient to determine v_1. The differential equation for v_1 follows by substituting the expansion (7) into the first of the field equations (1), noting that the incident wave satisfies it already, dividing by ε and letting ε tend to zero, yielding

$$(\nabla^2 + 1)v_1 = 0 \quad (-\infty < x < \infty, |y| > 0). \tag{17}$$

The general outwardly propagating solution of (17) subject to (15) and (16) is

$$v_1(x,y) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-n(k)|y|+ikx}}{k+(1+i\delta)\cos\theta_0}\left\{\left(\frac{1}{\mu}-1\right) i\sin\theta_0 \mathrm{sgn}(y) + \frac{\sin^2\theta_0+\mu(\cos^2\theta_0-\kappa^2)}{n(k)}\right\}\mathrm{d}k$$
$$+ \sum_{n=0}^{\infty} H_n^{(1)}(r)[A_n\sin(n\theta) + B_n\cos(n\theta)], \tag{18}$$

where $\mathrm{sgn}(y) = +1$ if $y > 0$ and -1 if $y < 0$, δ is a vanishingly small positive constant, $n(k) = (k^2-1)^{1/2}$, $r = (x^2+y^2)^{1/2}$, $\theta = \tan^{-1}(y/x)$, A_n and B_n are undetermined constants, and $H_n^{(1)}$ is the nth order Hankel function. Branch cuts in the complex k plane are taken so that $n(k) > 0$ when k is real and $|k| > 1$. The portion of (18) represented by the integral can be obtained by Fourier transformation as detailed in Appendix A; each term in the summation is an eigenfunction which is continuous across the line of the inclusion but singular at the origin.

Two important facts about (18) are that (a) further conditions are needed to determine A_n, B_n, $n = 0, 1, \ldots$, and (b) along with the associated expression for $\hat{w}_1(x,\tilde{y})$, it cannot satisfy the continuity conditions across B near the origin. Both points are manifestations of the singular nature of the perturbation, with the tip of the inclusion as the zone of non-uniformity. The tip is, in effect, outside the range of validity of the outer expansions (7) and (8). This in turn means that, although for $n \geq 1$ the r^{-n} singularity of $H_n^{(1)}(r)$ violates the requirement of bounded strain energy, the corresponding terms in the general expression (18) for v_1 cannot be excluded solely on this basis.

At this stage, we seek "inner expansions" in terms of stretched coordinates for the region near the tip. Thus with $x = \varepsilon\tilde{x}$, $y = \varepsilon\tilde{y}$, $V(x,y) = \tilde{V}(\tilde{x},\tilde{y})$, etc. eqns (1) and (2) become

$$(\tilde{\nabla}^2 + \varepsilon^2)\tilde{V} = 0 \quad [(\tilde{x},\tilde{y}) \in \tilde{M}], \qquad (\tilde{\nabla}^2 + \varepsilon^2\kappa^2)\tilde{W} = 0 \quad [(\tilde{x},\tilde{y}) \in \tilde{I}],$$
$$\tilde{V} = \tilde{W}, \qquad \partial\tilde{v}/\partial\tilde{n} = \mu\partial\tilde{w}/\partial\tilde{n} \qquad [(\tilde{x},\tilde{y}) \in \tilde{B}]. \tag{19}$$

The magnified geometry is shown in Fig. 2. The functions $\tilde{V}$ and $\tilde{W}$ will have "inner"

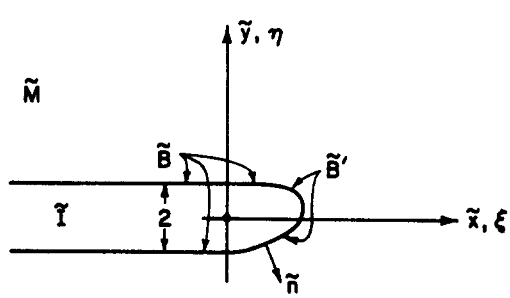

Fig. 2. Geometry of the near-tip region in the stretched coordinate system $\tilde{x} = x/\varepsilon$, $\tilde{y} = y/\varepsilon$. Here $\tilde{B}'$ refers to the non-parallel portions of the interface.

asymptotic expansions analogous to the outer expansions (7) and (8) for V and W. The inner and outer expansions must be related by a "matching principle." The principle will dictate the selection of gauge functions (functions of ε appearing as coefficients in the expansion) and will yield equations for various undetermined constants.

THE MATCHING PRINCIPLE

According to Fraenkel [9], a matching principle applicable when the gauge functions are either powers of ε or log ε or products of such powers can be stated as follows. Let $V^{(m)}(x,y)$ denote the function comprising all terms of the outer expansion with powers of ε up to m (and with any powers of log ε). This we designate as the outer expansion to order ε^m. Let $V^{(m,n)}$ denote the result of expressing $V^{(m)}$ in terms of $x = \varepsilon\bar{x}$, $y = \varepsilon\bar{y}$, expanding as $\varepsilon \to 0$, and retaining all terms with powers of ε up to n. This we designate the inner expansion to order ε^n of the outer expansion to order ε^m. With analogous definitions for $\bar{V}(\bar{x},\bar{y})$, Frankel's matching principle is

$$V^{(m,n)} = \bar{V}^{(n,m)} \tag{20}$$

for any choice of m and n, provided that the expression on either side of (20) is first rewritten in terms of the independent variables appearing on the other side.

The first consequence is that excepting B_0, all constants in the summation in (18) must vanish. If, for example the highest term present were $H_2^{(1)}(r)\cos(2\theta)$, then owing to the singularity of the Hankel function $V^{(1)}$ would contain a term of order $\varepsilon^{-1}\bar{r}^{-2}$, and would generate a term $V^{(1,-1)}$, of order $\varepsilon^{-1}\bar{r}^{-2}$ as ε tended to zero. The gauge function for the first term in the inner expansion would have to be ε^{-1}, and the spatially varying part would have to satisfy eqn (19) with $\varepsilon = 0$, and to be of order $\bar{r}^{-2}$ as $\bar{r}$ tended to infinity. Such a term would of necessity vanish identically. Thus the outer expansion $V^{(1)}$ to order ε is indeterminate only up to a constant B_0, and its inner expansion $V^{(1,1)}$ to order ε can be found from (7) and (18) to be

$$V^{(1,1)} = 1 - i\varepsilon(\bar{x}\cos\theta_0 + \bar{y}\sin\theta_0) + \frac{i\varepsilon}{\pi}\left[\frac{-\theta_0(\sin^2\theta_0 + \mu\cos^2\theta_0 - \mu\kappa^2)}{\sin\theta_0} + \left(1 - \frac{1}{\mu}\right)\theta\sin\theta_0\right]$$
$$+ \varepsilon B_0[1 + (2i/\pi)(\log\varepsilon + \log\bar{r} - \log 2 + \gamma)], \tag{21}$$

where γ is Euler's constant .577... The details are given in Appendix A.

The remainder of the analysis seeks to determine B_0 and to examine some features of the structure of the inner expansion. As to the form of the expansion, it can be seen from (21) that to match $V^{(1,1)}$ with $\bar{V}^{(1,1)}$ we must have

$$\bar{V}(\bar{x},\bar{y};\varepsilon) = \bar{v}_0(\bar{x},\bar{y}) + \varepsilon\bar{v}_1(\bar{x},\bar{y}) + \epsilon\log\epsilon\bar{v}_2(\bar{x},\bar{y}) + o(\epsilon\log\epsilon) \qquad [\epsilon \to 0,\ (\bar{x},\bar{y}) \text{ fixed}], \tag{22}$$

and analogously,

$$\bar{W}(\bar{x},\bar{y};\varepsilon) = \bar{w}_0(\bar{x},\bar{y}) + \varepsilon\bar{w}_1(\bar{x},\bar{y}) + \varepsilon\log\varepsilon\bar{w}_2(\bar{x},\bar{y}) + o(\varepsilon\log\varepsilon) \qquad [\varepsilon \to 0,\ (\bar{x},\bar{y}) \text{ fixed}]. \tag{23}$$

When (22) and (23) are inserted into the field equations (19), and ε repeatedly allowed to tend to zero, we find

$$\bar{\nabla}^2\bar{v}_k = 0 \qquad [(\bar{x},\bar{y}) \in \bar{M}], \qquad \bar{\nabla}^2\bar{w}_k = 0 \qquad [(\bar{x},\bar{y}) \in \bar{I}],$$
$$\bar{v}_k = \bar{w}_k,\ \partial\bar{v}_k/\partial\bar{n} = \mu\partial\bar{w}_k/\partial\bar{n} \qquad [(\bar{x},\bar{y}) \in \bar{B}];\ k = 0,1,2. \tag{24}$$

The matching principle (20) gives conditions at infinity to be satisfied by $\bar{v}_k$, $\bar{w}_k$. To satisfy it with $m = n = 0$ we must have

$$\lim_{\bar{r}\to\infty}\bar{v}_0(\bar{r},\theta) = \lim_{\varepsilon\to 0}\bar{v}_0(x/\varepsilon, y/\varepsilon) \equiv \bar{V}^{(0,0)} = V^{(0,0)} \equiv 1, \tag{25}$$

and the same limit for $\bar{w}_0$. (Here and henceforth the functions $\bar{v}_k$ are interchangeably regarded as dependent on either Cartesian or polar coordinates, the choice being indicated by the particular notation used.) In view of (25) and (24) with $k = 0$, we have by inspection the zero-order inner solution

$$\bar{v}_0(\bar{x},\bar{y}) \equiv 1 \quad [(\bar{x},\bar{y}) \in \bar{M}], \qquad \bar{w}_0(\bar{x},\bar{y}) \equiv 1 \; [(\bar{x},\bar{y}) \in \bar{I}]. \tag{26}$$

To satisfy (20) with $m = 0$ and $n = 1$ we require

$$1 + \lim_{\varepsilon\to 0}\{\varepsilon\bar{v}_1(x/\varepsilon, y/\varepsilon) + \varepsilon\log\varepsilon\,\bar{v}_2(x/\varepsilon, y/\varepsilon)\} \equiv \bar{V}^{(1,0)}$$

$$= V^{(0,1)} \equiv 1 - i(x\cos\theta_0 + y\sin\theta_0),$$

which implies that

$$\bar{v}_1(\bar{x},\bar{y}) = -\,i(\bar{x}\cos\theta_0 + \bar{y}\sin\theta_0) + o(\bar{r}/\log\bar{r}) \text{ as } \bar{r}\to\infty, \tag{27}$$

$$\bar{v}_2(\bar{x},\bar{y}) = o(\bar{r}/\log\bar{r}) \text{ as } \bar{r}\to\infty, \tag{28}$$

and the same limits for $\bar{w}_1$, $\bar{w}_2$. Again by inspection we see that (28) and (24) with $k = 2$ are solved by

$$\begin{aligned} \bar{v}_2(\bar{x},\bar{y}) &\equiv F \quad [(\bar{x},\bar{y}) \in \bar{M}], \\ \bar{w}_2(\bar{x},\bar{y}) &\equiv F \quad [(\bar{x},\bar{y}) \in \bar{I}], \end{aligned} \tag{29}$$

where F is an undetermined constant.

In summary, up to this point we have

$$\bar{V}^{(1)} = 1 + F\,\varepsilon\log\varepsilon + \varepsilon\bar{v}_1(\bar{x},\bar{y}).$$

With reference to (21), by applying the matching principle (20) with $m = n = 1$ we conclude that

$$\begin{aligned} \bar{v}_1(\bar{r},\theta) \sim -i(\bar{x}\cos\theta_0 + \bar{y}\sin\theta_0) + \frac{i}{\pi}\left[\frac{-\theta_0(\sin^2\theta_0 + \mu\cos^2\theta_0 - \mu\kappa^2)}{\sin\theta_0} + \left(1 - \frac{1}{\mu}\right)\sin\theta_0\right] \\ + B_0[1 + (2i/\pi)(\log\bar{r} - \log 2 + \gamma)] \text{ as } \bar{r}\to\infty, \end{aligned} \tag{30}$$

$$F = 2iB_0/\pi. \tag{31}$$

The asymptotic form (30), together with the field equations (24) for $k = 1$, provide sufficient conditions for the full determination of $\bar{v}_1$ (including B_0) and thus enable the solution to be completed to order ϵ, as outlined in the following section.

THE FIRST-ORDER INNER SOLUTION

The required form (30) contains terms of three distinct orders in $\bar{r}$ (viz. $\bar{r}^1$, $\log\bar{r}$ and $\bar{r}^0$), but only those of order $\bar{r}^1$ are fully specified. The equations governing $\bar{v}_1$ and $\bar{w}_1$ are therefore identical with those for the *static* antiplane deformation of an infinite solid containing a semi-infinite inclusion of width 2 and stiffness contrast μ, subjected to a uniform strain field at infinity. The strains at infinity are those corresponding to the displacements given by the first term on the r.h.s. of (30). This problem is also analogous to ones in heat conduction and electrostatics. However, none of these analogs seems to have been solved in a form useful to our purposes. Indeed, no closed-form solution appears to be possible, in contrast with the case studied by Crighton and Leppington [2]. They used conformal mapping, but here it fails, because a mapping of the exterior of the inclusion onto a half plane fails to map the interior onto the other half-plane.

The analysis will be facilitated by introducing a new function $\phi(\bar{x},\bar{y})$ defined by

$$\phi(\bar{x},\bar{y}) = \begin{cases} -i\bar{v}_1(\bar{x},\bar{y}) + \bar{x}\cos\theta_0 + \bar{y}\sin\theta_0 & [(\bar{x},\bar{y}) \in \bar{M}], \\ -i\bar{w}_1(\bar{x},\bar{y}) + \bar{x}\cos\theta_0 + \bar{y}\sin\theta_0 & [(\bar{x},\bar{y}) \in \bar{I}]. \end{cases} \quad (32)$$

From (32) and (24) with $k = 1$ we find that ϕ is governed by the field equation

$$\bar{\nabla}^2\phi = 0 \qquad [(\bar{x},\bar{y}) \in \bar{M} \cup \bar{I}] \quad (33)$$

and the continuity conditions

$$\phi^M = \phi^I, \qquad \frac{\partial\phi^M}{\partial\bar{n}} - \mu\frac{\partial\phi^I}{\partial\bar{n}} = (1-\mu)\frac{\partial}{\partial\bar{n}}(\bar{x}\cos\theta_0 + \bar{y}\sin\theta_0) \qquad [(\bar{x},\bar{y}) \in \bar{B}], \quad (34)$$

where the superscripts M and I indicate that the boundary $\bar{B}$ is approached from the matrix or inclusion side, respectively (see Fig. 2). Moreover, comparison of (32) with (30) shows that

$$\phi = o(\log\bar{r}) \text{ as } \bar{r} \to \infty, \quad (35)$$

and that once the coefficient of the logarithmetic term is determined, so is B_0.

Although it is impossible to solve explicitly for ϕ, its dominent terms at infinity can be found as follows. Let

$$f(s) = \frac{\partial\phi^M}{\partial\bar{n}} - \frac{\partial\phi^I}{\partial\bar{n}}, \quad (36)$$

where s is arc length along $\bar{B}$. Then any function satisfying (33), the first of (34), and (35–36) must take the form

$$\phi(\bar{x},\bar{y}) = \frac{1}{2\pi}\int_{\bar{B}} f(s)\log\{[\bar{x} - \xi(s)]^2 + [\bar{y} - \eta(s)]^2\}^{1/2}\,\mathrm{d}s + G \quad (37)$$

where (ξ,η) are coordinates of points on $\bar{B}$ and G is an arbitrary constant. By using (37) as detailed in Appendix B, the asymptotic expansion of ϕ to order $\bar{r}^0$ can be found; when compared with (32) it yields

$$\begin{aligned} \bar{v}_1(\bar{x},\bar{y}) = {} & -i(\bar{x}\cos\theta_0 + \bar{y}\sin\theta_0) + i(1-\mu)\cos\theta_0\log\bar{r}/\pi \\ & + i(\theta/\pi)(1 - 1/\mu)\sin\theta_0 + iG + o(1) \text{ as } \bar{r} \to \infty, -\pi < \theta < \pi. \end{aligned} \quad (38)$$

The *derived* form (38) must agree with the *required* form (30). Comparison of the logarithmic terms yields

$$B_0 = (1-\mu)(\cos\theta_0)/2, \quad (39)$$

and of the terms of order $\bar{r}^0$ yields a value for G. The θ-dependent terms already agree, thereby providing a consistency check. Equation (31) gives F, and both the inner and outer solutions are now complete to order ε.

Because B_0 is evidently independent of the shape of tip of inclusion, and contains to first order all the information transmitted from the inner to outer regions, we may conclude that the outer expansion is, to first order, independent of the shape of tip of the inclusion.

Each term of (38) has a physical basis when $\bar{v}_1$ is interpreted as the static displacement field in an infinite matrix uniformly strained at infinity. The first term is the displacememt field which causes the strain at infinity. When this is subtracted away, the rest of the field, which is just $i\phi$, satisfies conditions proportional to (33)–(35). In particular, the right hand side of the second of (34) can be regarded as a line of body forces applied along the interface $\bar{B}$. The forces on the parallel segments of $\tilde{B}$ will be equal and opposite, and far from the origin will cause a uniform dislocation of the upper segment with respect to the lower one. The corresponding field in the matrix is approximated by the θ-dependent term in (38), which suffers precisely the correct dislocation from $\theta = -\pi$ to π. The body forces on the rest of $\tilde{B}$, the segment near the origin denoted $\tilde{B}'$, will have a non-zero resultant and, at great distance from the origin, will have the same effect as a concentrated force, thereby giving rise to the logarithmic term in (38).

THE FAR FIELD

In this section we derive a uniformly valid asymptotic expansion of the scattered field far from the tip of the inclusion. The method to be used is basically that of steepest descents; however, a modification proposed by Van der Waerden[10] and exemplified by the work of Bazer and Karp[11] must be incorporated so as to render the expansion *uniformly* valid. The rays of potential difficulty are those at $\theta = \pm(\pi - \theta_0)$, which bound the zones containing the geometrically reflected and transmitted plane waves, respectively. These waves are represented by poles in a complex transform plane which contribute if they are crossed in deforming the original integration contour to that of the steepest descents. When $\theta \to \pm(\pi - \theta_0)$, one of the poles and the saddle point coalesce, and the usual saddle point expansion becomes singular. The proposed modification analytically isolates the pole such that the part containing it can be evaluated explicitly, and the remaining integral can be uniformly approximated by a standard saddle point expansion.

To first order in ε, the scattered field is just εv_1, where v_1 comprises the integral in (18), which we denote as I, plus the term $B_0 H_0^{(1)}(r)$, with B_0 given by (39). The variable change $k = \cos\beta$ converts the integral to the more convenient form

$$I = \frac{1}{2\pi}\int_{C_\beta} \frac{P\sin\beta\,\mathrm{sgn}\theta - Q}{\cos\beta + \cos\theta_0}\, \mathrm{e}^{ir\cos(\beta - |\theta|)}\, \mathrm{d}\beta \tag{40}$$

where $P = \sin\theta_0(1 - 1/\mu)$, $Q = \sin^2\theta_0 + \mu(\cos^2\theta_0 - \kappa^2)$, θ ranges from $-\pi$ to π, and Fig. 3 depicts the contour C_β. The pole giving rise to the reflected and transmitted plane waves is at $\beta = \theta_r \equiv \pi - \theta_0$. Note that the branch cuts needed in the k-plane have been eliminated by the change of variable.

The saddle point occurs at $\beta = |\theta|$, where the derivative with respect to β of the exponent in (40) vanishes. The contour of steepest descents is then determined by the condition $i\cos(\beta - |\theta|) = i - s^2$, where s^2 is a *real* variable ranging over $(-\infty, \infty)$. When solved for s this becomes

$$s = -\mathrm{e}^{i\pi/4}\sqrt{2}\sin\left[\tfrac{1}{2}(\beta - |\theta|)\right], \tag{41}$$

the sign being chosen so that as s increases from $-\infty$ to ∞, the steepest descents contour C_s parameterized by (41) is traversed in the same general direction as the original contour C_β (see Fig. 3).

We now deform the contour C_β to C_s and note that in doing so the pole at $\beta = \pi - \theta_0 \equiv \theta_r$ will be crossed if $|\theta| > \theta_r$. With (41) implicitly giving β in terms of s, from (40) there follows

$$I = \frac{\exp(ir)}{2\pi}\int_{-\infty}^{\infty}\left[\frac{P\sin\beta(s)\mathrm{sgn}\theta - Q}{\cos\beta(s) + \cos\theta_0}\right]\frac{\exp(-rs^2)}{\mathrm{d}s/\mathrm{d}\beta}\,\mathrm{d}s$$
$$+ iH(|\theta| - \theta_r)(P\,\mathrm{sgn}\theta - Q/\sin\theta_0)\exp[ir\cos(|\theta| - \theta_r)] \tag{42}$$

The *standard* saddle point expansion could now be obtained by replacing the coefficient of the exponential in the integrand by its value at $s = 0$. Instead, the pole at $\beta(s) = \theta_r = \pi - \theta_0$ or [see (41)]

$$s = s_r \equiv -\mathrm{e}^{i\pi/4}\sqrt{2}\cos\left[\tfrac{1}{2}(|\theta| + \theta_0)\right]$$

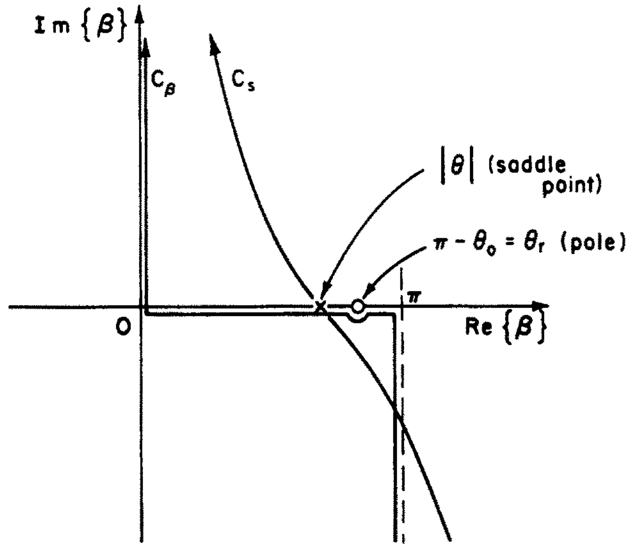

Fig. 3. Integration contours and special points in the complex β plane.

will be analytically isolated as follows. First, by multiplying the integrand by

$$1 = -e^{i\pi/4}\sqrt{2}\{\sin[\tfrac{1}{2}(\beta - |\theta|)]\} - \cos[\tfrac{1}{2}(|\theta| + \theta_0)]/(s - s_r)$$

and using (41) to calculate $ds/d\beta$, the first term of (42) becomes

$$I' = \int_{-\infty}^{\infty} h(s)\frac{\exp(-rs^2)}{s - s_r}\,ds \tag{43}$$

where

$$h(s) = \frac{\exp(ir)(P\sin\beta\,\mathrm{sgn}\,\theta - Q)\{\sin[\frac{1}{2}(\beta - |\theta|)] - \cos[\frac{1}{2}(|\theta| + \theta_0)]\}}{\pi(\cos\beta + \cos\theta_0)\cos[\frac{1}{2}(\beta - |\theta|)]}. \tag{44}$$

Next, (43) is rewritten in the equivalent form

$$I' = \int_{-\infty}^{\infty}\left[\frac{sh(s_r)}{s_r(s - s_r)} + \frac{h(s) - h(s_r)}{s - s_r} - \frac{h(s_r)}{s_r}\right]e^{-rs^2}\,ds. \tag{45}$$

The first term can be evaluated explicitly[10]:

$$\int_{-\infty}^{\infty}\frac{sh(s_r)}{s_r(s - s_r)}e^{-rs^2}\,ds = \frac{h(s_r)}{s_r}\left(\frac{\pi}{r}\right)^{\frac{1}{2}} + \frac{s_r h(s_r)\exp(-rs_r^2 + i\pi/4)\,\mathrm{erfc}\{(1 - i)r^{\frac{1}{2}}\cos[\frac{1}{2}(|\theta| + \theta_0)]\}}{\sqrt{2}|\cos[\frac{1}{2}(|\theta| + \theta_0)]|}. \tag{46}$$

The latter two terms of (45) can be approximated by the usual saddle point expansion, the first term of which follows by setting $s = 0$ in the coefficient of the exponential:

$$\int_{-\infty}^{\infty}\left[\frac{h(s) - h(s_r)}{s - s_r} - \frac{h(s_r)}{s_r}\right]e^{-rs^2}\,ds \sim -\frac{h(0)}{s_r}\left(\frac{\pi}{r}\right)^{\frac{1}{2}}. \tag{47}$$

Finally, by combining eqns (18), (39) and (42–47), and using the standard asymptotoc expansion of $H_0^{(1)}(r)$ [12], the far field expansion of $v_1(r)$ may be written

$$v_1 \sim R(\theta)r^{-1/2}\exp(ir) + i(P\,\mathrm{sgn}\,\theta - Q/\sin\theta_0)\exp[ir\cos(|\theta| - \theta_r)] \times \left\{H(|\theta| - \theta_r) - \tfrac{1}{2}\mathrm{sgn}(|\theta| - \theta_r)\mathrm{erfc}\left[(1-i)r^{\frac{1}{2}}|\cos(\tfrac{1}{2}\{|\theta| + \theta_0\})|\right]\right\} \tag{48}$$

where

$$R(\theta) = \frac{\exp(-i\pi/4)}{(2\pi)^{1/2}(\cos\theta + \cos\theta_0)}\left\{(\frac{1}{\mu} - 1)\sin\theta\sin\theta_0 + 1 - \mu\kappa^2 + (1-\mu)\cos\theta\cos\theta_0 + \cos[\tfrac{1}{2}(|\theta| - \theta_0)][(1 - 1/\mu)\mathrm{sgn}\,\theta\sin\theta_0 - \sin\theta_0 - \mu(\cos^2\theta_0 - \kappa^2)/\sin\theta_0]\right\}. \tag{49}$$

Equation (48) is uniformly valid as r tends to infinity. If θ remains fixed and not equal to $\pm\theta_r$, expansion of the complementary error function for large argument reduces (48) to

$$v_1 \sim \frac{\exp(ir - i\pi/4)}{(2\pi r)^{1/2}(\cos\theta + \cos\theta_0)}\left[\left(\frac{1}{\mu} - 1\right)\sin\theta\sin\theta_0 + 1 - \mu\kappa^2 + (1-\mu)\cos\theta\cos\theta_0\right] + i(P\,\mathrm{sgn}\,\theta - Q/\sin\theta_0)\exp[ir\cos(|\theta| - \theta_r)]H(|\theta| - \theta_r) \text{ as } r\to\infty,\ \theta \text{ fixed, } \theta \neq \theta_r. \tag{50}$$

This is precisely the non-uniform expression which would have resulted from an elementary saddle-point expansion of the integral in (42). The second term represents non-decaying plane waves. The reflected wave, appearing when $\theta > \theta_r$, may be written as $A^{(r)}\exp[ir\cos(\theta - \theta_r)]$, where the *reflection coefficient* $A^{(r)}$ is given by

$$A^{(r)} = -\varepsilon i[\sin\theta_0/\mu + \mu(\cos^2\theta_0 - \kappa^2)/\sin\theta_0] + o(\varepsilon) \text{ as } \varepsilon\to 0.$$

The transmitted plane wave appears for $\theta < -\theta_r$, and may be written as $A^{(t)}\exp[ir\cos(\theta + \theta_r)]$, where

$$A^{(t)} = 1 + \varepsilon i[(1/\mu - 2)\sin\theta_0 + \mu(\kappa^2 - \cos^2\theta_0)/\sin\theta_0] + o(\varepsilon) \text{ as } \varepsilon\to 0.$$

(The first term is the contribution of the incident wave.)

The remaining terms in (50), being of order $r^{1/2}$, comprimise the radiated wave. It is interesting to note that these terms are symmetric in θ and θ_0. De Hoop [13] proved that such reciprocity is necessary for scattering of plane electromagnetic waves by *bounded* bodies. The proof does pertain to the scalar wave equation as a special case, but does not carry over to scatterers of infinite extent. While it is unlikely that the observed reciprocity is merely fortuitous, a proof of its necessity does not seem to be available.

The situation in the far field at the singular rays $\theta = \pm\theta_r$ can be inferred directly from the expansion (48). The argument of the complementary error function vanishes there, so because $\mathrm{erfc}(0) = 1$ and $H(z) - \frac{1}{2}\mathrm{sgn}(z) = \frac{1}{2}$, the non-decaying portion is just half of the basic plane wave contribution of the type discussed earlier. The amplitude $R(\theta)$ of the decaying portion remains bounded as $\theta \to \pm\theta_r$, as follows by applying L'Hospital's rule to (49).

DISCUSSION

The first order power flux P_1 may be defined as

$$P_1 = \tfrac{1}{2}Im\{v_1(\partial v_1/\partial r)^*\}, \tag{51}$$

where $(\ldots)^*$ denotes the complex conjugate. The quantity $\varepsilon^2 P_1 r\,d\theta$ would then represent a first approximation to the normalized temporal average, over a cycle, of the scattered energy transmitted across the segment $r\,d\theta$. As r tends to infinity, the uniformly valid expression (48) may be used to compute P_1. However, because for fixed θ P_1 does not decay when $|\theta| > \theta_r$ but decays as r^{-1} when $|\theta| < \theta_r$ a specific value of r must be chosen for examination of the full scattered field. Thus for $r = 1000$, polar plots will be given for $P_{1,db}$, where

$$P_{1,db} = 10\log_{10}(P_1).$$

Each plot will be for specific values of μ and κ; however, it is somewhat more convenient to take μ and δ as parameters, where $\delta = \mu\kappa^2$ is the mass density ratio of the inclusion to the matrix.

Figure 4 shows how P_1 varies with δ for $\mu = 1.5$ and $\theta_0 = 45°$. The relatively large values of P_1 for $|\theta| > 135°$ correspond to the reflected and transmitted plane waves as given by the second term of (50). The smaller values for $|\theta| < 135°$ correspond to the decaying, radiated wave given by the first term of (50). The smooth transition between these regimes is provided by the uniformly valid expansion (48) and could not have been inferred from the simpler expression (50). Also, the oscillations in intensity near $\theta = \pm 135°$ are characteristics of the more elaborate expansion (48) not present in (50).

A most striking feature of Fig. 4 is the appearance of either one or two sharp dips in P_1 for certain values of δ. These are associated with nulls in the radiated wave as given by the first term of (50). Indeed, the quantity in brackets vanishes when

$$\cos\theta = \frac{-\mu\zeta\cos\theta_0 \pm \sin\theta_0(\mu^2\cos^2\theta_0 + \sin^2\theta_0 - \zeta^2)^{\frac{1}{2}}}{\mu^2\cos^2\theta_0 + \sin^2\theta_0}, \tag{52}$$

where $\zeta = \mu(1-\delta)/(1-\mu)$. The corresponding value of $\sin\theta$ is

$$\sin\theta = -(\zeta + \mu\cos\theta\cos\theta_0)/\sin\theta_0. \tag{53}$$

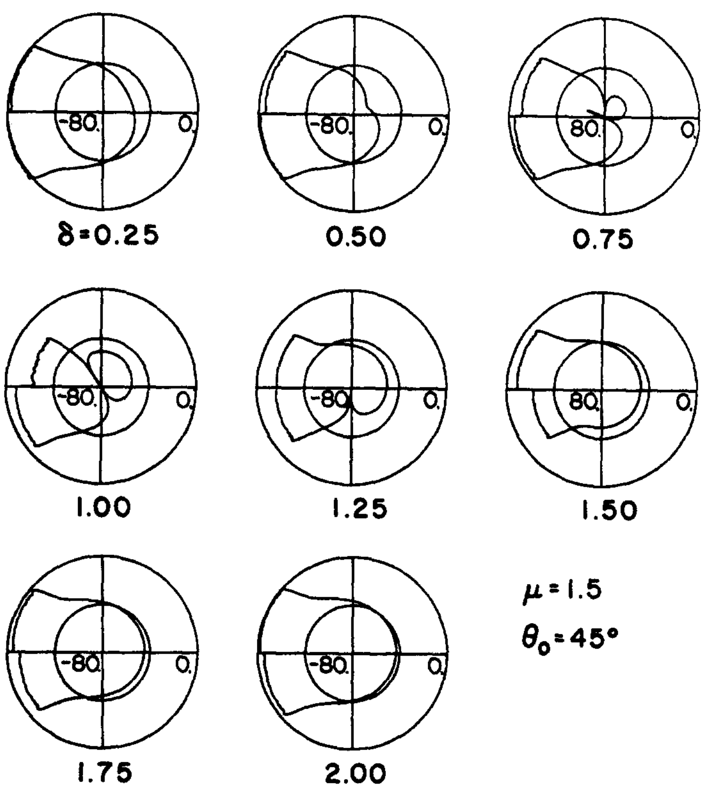

Fig. 4. Polar plots of the first order scattered power flux $P_{1,db}$ for stiffness ratio $\mu = 1.5$, angle of incidence $\theta_0 = 45°$, $r = 1000$ and a range of mass density ratios δ. The approximate non-dimensionalized scattered power flux is $\epsilon^2 P_1$.

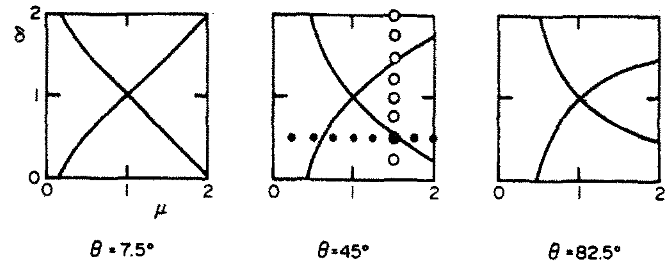

Fig. 5. Material parameters for which the radiated wave has nulls. Points in the shaded regions represent pairs of stiffness ratio μ and density ratio δ for which nulls exist. Open circles denote the cases shown in Fig. 4; closed circles, Fig. 6.

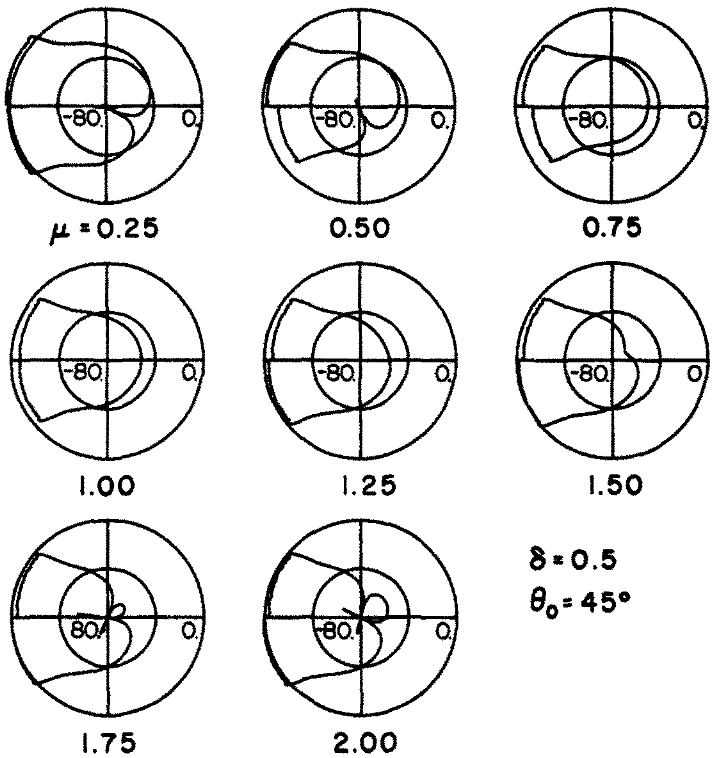

Fig. 6. Polar plot of the first order scattered power flux $P_{1,db}$ for mass density ratio $\delta = 0.5$, incident angle $\theta_0 = 45°$, $r = 1000$ and a range of stiffness ratios μ.

These will be real if

$$|1-\delta| \leq |1-\mu|(\cos^2\theta_0 + \sin^2\theta_0/\mu^2)^{\frac{1}{2}}, \tag{54}$$

and if real, are always less than unity in absolute value, thus yielding angles θ_1, θ_2 where the radiated flux does not vanish. In Fig. 5, for $\theta_0 = 7.5°$, 45° and 82.5°, the regions in $\mu - \delta$ space satisfying (54) are shaded. For $\theta_0 = 45°$, the open circles represent the plots of Fig. 4. Thus, for $\mu = 1.5$, as δ increases we progress from the regime where nulls are absent, to one where they are present, and again to one where they are absent. If nulls are present according to (54), but one of them falls within the plane-wave zone $|\theta| > \theta_r$, it will be obscured by the plane wave. Such is the case in Fig. 4 for $\delta = 1.25$.

From (52) and (53) we may show that

$$\tan(\theta_1 + \theta_2) = 2\mu\cos\theta_0\sin\theta_0/(\mu^2\cos^2\theta_0 - \sin^2\theta_0)$$

where the sign of $\theta_1 + \theta_2$ must coincide with that of the numerator. Thus the angle $(\theta_1 + \theta_2)/2$ bisecting that between nulls is independent of the density ratio δ, and depends only on the stiffness ratio μ and angle of incidence θ_0 (One might suppose that $\theta_1 - \theta_2$ would be independent of μ, but this is not the case.)

Figure 6 shows how P_1 varies with μ for $\delta = 0.5$ and $\theta_0 = 45°$. The corresponding points are indicated by solid circles in Fig. 5. Note that the pattern is symmetric for $\mu = 1$, as is always the case when the stiffness contrast vanishes.

A glance at Fig. 5 indicates that with θ_0 and either μ or δ fixed, as the other is varied from zero upward, radiation patterns both with and without nulls will be observed. If however, μ and δ are fixed and θ_0 is varied, both types of patterns will not necessarily be observed. For δ close to unity and μ sufficiently different from unity, only patterns with nulls will occur, while in the converse case, nulls will be absent for all incident angles between 0° and 90°.

CONCLUSIONS

The method of singular perturbations and matched asymptotic expansions leads to closed-form solutions for the field of *SH* waves scattered by a thin, semi-infinite inclusion. The explicit form of the far field follows by a modification of the saddle point method. The scattered field comprises non-decaying, reflected and transmitted plane waves which are dominant along rays whose angular separation from the inclusion is less than the incident angle, and a radiated wave which decays as $r^{1/2}$. The latter may have either two nulls or none, depending on the incident angle and the stiffness and density contrast. When nulls are present, the bisector of the angle between them is independent of the density contrast.

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APPENDIX A

The first order outer solution

Here we derive by Fourier transforms the solution of the wave equation (17) subject to the discontinuity conditions (15) and (16). It is convenient to regard the solution as a sum of a part $v_1^{(s)}$ which is symmetric about $y = 0$, and an anti-symmetric part $v_1^{(a)}$, and then to solve for each only in $y > 0$. The symmetric part will contain the jump in normal derivative $\partial v_1/\partial y$ prescribed by 16—vanishing for $x > 0$—and no jump in $v_1^{(s)}$. The antisymmetric part will have a gradient with the same limit from above or below the axis $y = 0$, but a jump in $v_1^{(a)}$—again vanishing for $x > 0$—equal to that given by (15)†. From (15) to (17) then, the differential equations and boundary conditions are

$$(\nabla^2 + 1)v_1^{(p)} = 0, \qquad p = a,s, \tag{55}$$

†Here the distinctions among thin cavities, rigid inclusions, and elastic inclusions are evident. The former two cases would require specification of either the normal derivative itself (for a cavity) or the displacement itself (for a rigid inclusion) on *both* sides of the scatterer, while for the latter the *jump* across the inclusion in both the displacement and its normal derivative are specified, but the displacement and normal derivative themselves are left unspecified. The former two cases, when formulated in half-planes, lead to mixed boundary values problems, while the latter, as shown in the text, leads to simple, non-mixed problems.

$$\partial v_1^{(s)}/\partial y\bigg|_{y=0^+} = [\sin^2\theta_0 + \mu(\cos^2\theta_0 - \kappa^2)]\exp[-ix\cos\theta_0(1+i\delta)]H(-x) \tag{56}$$

$$v_1^{(a)}(x,0^+) = i(1-1/\mu)\sin\theta_0\exp[-ix\cos\theta_0(1+i\delta)]H(-x) \tag{57}$$

where the small positive constant δ (which renders the medium slightly dissipative) serves merely to place the poles on the proper side of the inversion contour in the transform plane.

The Fourier transform and its inverse are here defined by

$$\bar{f}(k,y) = \int_{-\infty}^{\infty} f(x,y)\exp(-ikx)\,dx, \quad f(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(k,y)\exp(ikx)\,dk \tag{58}$$

for an arbitrary function $f(x,y)$. When the transform operator is applied to (55) there results

$$\left(\frac{d^2}{dy^2} + n^2\right)\bar{v}_1^{(p)} = 0, \qquad p = a,s, \tag{59}$$

where $n^2 = k^2 - 1$. With branch cuts for $n(k)$ defined such that $Re\{n\} \geq 0$, the solutions of (59) which remain bounded in $y \geq 0$ are

$$\bar{v}_1^{(p)}(k,y) = A^{(p)}(k)\exp(-ny), \qquad p = a,s. \tag{60}$$

Application of the transform operator to (56), (57) yields

$$\partial\bar{v}_1^{(s)}/\partial y\bigg|_{y=0^+} = i[\sin^2\theta_0 + \mu(\cos^2\theta_0 - \kappa^2)]/[\cos\theta_0(1+i\delta) + k], \tag{61}$$

$$\bar{v}_1^{(a)}(k,0^+) = (1/\mu - 1)\sin\theta_0(1+i\delta) + k]. \tag{62}$$

By comparison with the expressions

$$\partial\bar{v}_1^{(s)}/\partial y\bigg|_{y=0^+} = -n(k)A^{(s)}(k), \quad \bar{v}_1^{(a)}(k,0^+) = A^{(a)}(k),$$

[which follow from (60)], (61) and (62) yield expressions for $A^{(s)}$, $A^{(a)}$. Substitution into (60) and then the second of (58) yields

$$v_1^{(s)} = \frac{\sin^2\theta_0 + \mu(\cos^2\theta_0 - k^2)}{2\pi i}\int_{-\infty}^{\infty} \frac{\exp[-n(k)y + ikx]\,dk}{n(k)[k + (1+i\delta)\cos\theta_0]}, \quad y > 0; \tag{63}$$

$$v_1^{(a)} = \frac{(1/\mu - 1)\sin\theta_0}{2\pi}\int_{-\infty}^{\infty} \frac{\exp[-n(k)y + ikx]\,dk}{k + (1+i\delta)\cos\theta_0}, \quad y > 0. \tag{64}$$

The values for $y < 0$ follow from these by symmetric or antisymmetric extensions respectively, and the extended expressions combine to yield the integral in (18).

To calculate $V^{(1,1)}$ as in (21), it is necessary to find the limiting forms of (63), (64) for small x,y. Regarding (63), one may simply set $x = y = 0$ and analyze the integral

$$I_s = \int_1^{\infty} \frac{dk}{(k)[k + (1+i\delta)\cos\theta_0]}.$$

By deforming the branch cut from $k = 1$ to lie along the positive real axis, and then deforming the contour to wrap around this branch cut, the pole at $k = -(1+i\delta)\cos\theta_0$ is *not* crossed and the integral becomes

$$I_s = 2\int_1^{\infty} \frac{dk}{(k^2-1)^{1/2}(k + \cos\theta_0)},$$

where δ has been set to zero since it has served its purpose. The variable change $\alpha = k + \cos\theta_0$ converts this to the form (380.111) of [13], which yields the value

$$I_s = 2\theta_0/\sin\theta_0. \tag{65}$$

If the integral in (64) is denoted I_a, we note that for $y > 0$,

$$\frac{\partial I_a}{\partial x} + i\cos\theta_0 I_a = i\int_{-\infty}^{\infty} \exp(-ny + ikx)\,dk = -i\frac{\partial}{\partial y}\int_{-\infty}^{\infty} \frac{\exp(-ny + ikx)}{n}\,dk = \pi\frac{\partial}{\partial y}H_0^{(1)}(r) = -\pi\sin\theta H_1^{(1)}(r). \tag{66}$$

The third equality follows from eqns (7.85) and (7.97) of [14], and the fourth from the chain rule and eqns (9.13) and (9.1.28) of [12]. Next, integration of (66) yields

$$I_a = \pi\exp(-ix\cos\theta_0)\int_x^{\infty}\sin[\theta(\xi,y)]\exp(i\xi\cos\theta_0)H_1^{(1)}[(\xi^2+y^2)^{\frac{1}{2}}]\,d\xi. \tag{67}$$

By using the identity $\sin[\theta(\xi,y)] = y(\xi^2+y^2)^{-\frac{1}{2}}$ and the variable change $\xi = r\eta + x$, (67) becomes $+\,1)^{-\frac{1}{2}}\,d\eta$.

$$I_a = \pi r\sin\theta\int_0^{\infty}\exp(ir\eta\cos\theta_0)H_1^{(1)}[r(\eta^2+2\eta\cos\theta+1)^{\frac{1}{2}}](\eta^2+2\eta\cos\theta+1)^{-\frac{1}{2}}\,d\eta. \tag{68}$$

The form of I_a as r trends to zero follows by expanding the integrand for small r. Because $H_1^{(1)}(z) \sim -2i/\pi z$ (see eqn (9.1.9) of [12]) we have

$$I_a \sim -2i\sin\theta\int_0^{\infty}(\eta^2+2\eta\cos\theta+1)^{-1}\,d\eta = -2i\theta \text{ as } r\to 0, \tag{69}$$

where the integral is calculated from eqn (857.01) of [13].

Substitution of (65) and (69) into (63) and (64) yields expressions which when multiplied by ε combine to give the third term on the r.h.s. of (21). The last term follows from the expansion—for small r—of $H_0^{(1)}(r)$ (see eqns (9.1.3), (9.1.10) and (9.1.11) of [12]).

APPENDIX B

The far field of the first order inner solution

Here we expand the integral in (37) to order $\bar{r}^0$ as $\bar{r}\to\infty$. From (34) and (36) there follows

$$f(s) = (1-\mu)\frac{\partial}{\partial\bar{n}}(\xi\cos\theta_0+\eta\sin\theta_0-\phi'). \tag{70}$$

By factoring out $\bar{r} = (\bar{x}^2+\bar{y}^2)^{\frac{1}{2}}$ from the integrand, (37) becomes

$$2\pi(\phi - G) = (1-\mu)\log\bar{r}\int_{\bar{B}}\frac{\partial}{\partial\bar{n}}(\xi\cos\theta_0+\eta\sin\theta_0)\,ds - (1-\mu)\log\bar{r}\int_{\bar{B}}\frac{\partial\phi'}{\partial\bar{n}}\,ds$$
$$+(1-\mu)\int_{\bar{B}}\log[(\cos\theta-\xi/\bar{r})^{\frac{1}{2}}+(\sin\theta-\eta/\bar{r})^{\frac{1}{2}}]\frac{\partial}{\partial\bar{n}}(\xi\cos\theta_0+\eta\sin\theta_0-\phi')\,ds. \tag{71}$$

The first integral in (71) is evaluated by first noting that on the parallel parts of $\bar{B}$ at $\bar{y} = \pm 1$, we have $\bar{n} = \pm\eta$, $\partial\xi/\partial\bar{n} = 0$, and $\partial\eta/\partial\bar{n} = \pm 1$ respectively (see Fig. 2). There is therefore no contribution from corresponding parallel segments. For the remaining part $\bar{B}'$ near the tip, the identities $\partial\xi/\partial\bar{n} = \partial\eta/\partial s$, $\partial\eta/\partial\bar{n} = -\partial\xi/\partial s$ convert it to

$$\int_{\bar{B}'}(\cos\theta_0\,d\eta - \sin\theta_0\,d\xi) = 2\cos\theta_0, \tag{72}$$

since η increases by 2 and ξ returns to its original value.

The second integral in (71) vanishes, as follows by closing the contour $\bar{B}$ by a small segment at $\eta = -\infty$, applying Green's theorem, and noting that ϕ is harmonic in the inclusion.

The third integral in (71) cannot be evaluated explicitly. By considering separately the parallel parts of $\bar{B}$ and the part near the end, the latter is seen to vanish as $\bar{r}\to\infty$ and the integral may be written as

$$\int_{-\infty}^{0}\left\{\left[\sin\theta_0-\left.\frac{\partial\phi'}{\partial\bar{y}}\right|_{\bar{y}=1}\right]\log\left[\left(1-\frac{2\xi\cos\theta_0+2\sin\theta_0}{\bar{r}}+\frac{\xi^2+1}{\bar{r}^2}\right)^{1/2}\right]\right.$$
$$\left.-\left[\sin\theta_0-\left.\frac{\partial\phi'}{\partial\bar{y}}\right|_{\bar{y}=-1}\right]\log\left[\left(1-\frac{2\xi\cos\theta_0-2\sin\theta_0}{\bar{r}}+\frac{\xi^2+1}{\bar{r}^2}\right)^{\frac{1}{2}}\right]\right\}d\xi + o(\bar{r}) \text{ as } \bar{r}\to\infty.$$

Now the variable change $\xi = \bar{r}\zeta$ is introduced, leaving the integration limit unchanged. As $\bar{r}\to\infty$, arguments of the terms $\partial\phi'/\partial\bar{y}$ tend to $-\infty$, where both take the common constant value $(\mu-1)(\sin\theta_0)/\mu$ (see discussion following eqn 39). The logarithmic terms then combine, yielding the form

$$-\frac{2\sin^2\theta_0}{\mu}\int_{-\infty}^{0}\frac{d\xi}{1-2\xi\cos\theta_0+\xi^2} = -\frac{2\theta\sin\theta_0}{\mu} \tag{73}$$

where the integral is evaluated from eqn (160.01) of [13]. Thus with (72) for the first integral in (71), zero for the second, and (73) for the third, the outer expansion to order $\bar{r}^0$ of the first order inner expansion is obtained for use in eqn (38).